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Concrete examples of operator monotone functions obtained by only applying Löwner-Heinz inequality

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§1. Introduction

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T\geq 0$) if $(Tx, x)\geq 0$ for all $x\in H$ and also an operator T is said to be strictly positive (denoted by $T>0$) if T is positive and invertible. A real valued continuous function $f(t)$ on $(0, \infty)$ is said to be *operator monotone* if $f(A)\geq f(B)$ holds for any $A\geq B$.

K. Löwner [10] had established the deep theory on operator monotone functions and also he had given a definitive characterization of operator monotone functions as follows.

Theorem L (K. Löwner.) *A function f: $(0, \infty)$ is operator monotone in $(0, \infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_{0}^{\infty}\frac{t}{t+s}dm(s)$$

with $a\in \mathbb{R}$ and $b\geq 0$ and a positive measure m on $(0, \infty)$ such that

$$\int_{0}^{\infty}\frac{dm(s)}{1+s}<+\infty.$$

Next we state the Löwner-Heinz inequality which is quite useful tool in this paper.

Theorem LH (Löwner-Heinz inequality).

(LH) t^{α} *is an operator monotone function for any* $\alpha\in[0,1]$.

Let $\alpha_{j}, \beta_{j}, \gamma_{j}, \ldots\in[0,1]$ *for* $j=1,2, \ldots, n$. *Then the following* (LH-1) *and* (LH-2) *are immediate consequences of* (LH).

(LH-1) $\left(\frac{1}{t^{\alpha_{1}}+..+t^{\alpha_{n}}}+\frac{1}{t^{\beta_{1}}+..+t^{\beta_{n}}}+\frac{1}{t^{\gamma_{1}}+..+t^{\gamma_{n}}}+\ldots\right)^{-1}$ *is an operator monotone function, in particular,* $(t^{-\alpha_{1}}+t^{-\alpha_{2}}+\ldots+t^{-\alpha_{n}})^{-1}$ *is an operator monotone function.*

(LH-2) $(1+t^{-1})^{-\alpha_{1}}+(1+t^{-1})^{-\alpha_{2}}+\ldots+(1+t^{-1})^{-\alpha_{n}}$ *is an operator monotone function.*

Although (LH) of the Theorem LH was originally proved by Theorem L [10] and secondly by Heinz [6], Pedersen [11] gave an elegant proof of Theorem LH without appealing to Theorem L and also Bhatia [2, Theorem V.1.9] has given a nice different proof of Theorem LH without appealing to Theorem L (also see Bhatia [3, Theorem 4.2.1]).

In this short paper, we study concrete examples of operator monotone functions obtained by only applying Theorem LH without appealing to Theorem L and also we give an elementary proof of Theorem A ([4][7]) stated in §3 by only applying Theorem LH.

We state the following obvious result.

Lemma 1.

(1.1) *If $T \geq 0$, then $T^{\frac{k}{n}} - I = (T^{\frac{1}{n}} - I)(T^{\frac{k-1}{n}} + T^{\frac{k-2}{n}} + \ldots + T^{\frac{1}{n}} + I)$*

for any natural number n and k such that $1 \leq k \leq n$, in particular

If $T \geq 0$, then $T - I = (T^{\frac{1}{n}} - I)(T^{1-\frac{1}{n}} + T^{1-\frac{2}{n}} + \ldots + T^{\frac{1}{n}} + I)$

for any natural number n.

(1.2) $\lim_{n\to\infty} n(T^{\frac{1}{n}} - I) = \log T$ *holds for any $T > 0$.*

§2. **Concrete examples of operator monotone functions derived from $\lim_{n\to\infty} n(T^{\frac{1}{n}} - I) = \log T$ and Löwner-Heinz inequality**

Theorem 2.1.

(i) $f(t) = \dfrac{1}{(1+t)\log(1+\frac{1}{t})}$ *is an operator monotone function.*

(ii) $g(t) = t(1+t)\log(1+\dfrac{1}{t})$ *is an operator monotone function.*

Proof. Let $A \geq B > 0$.

(i). We have only to show the following (2.1) on order to (i)

(2.1) $$\frac{I}{(I+A)\log(I+A^{-1})} \geq \frac{I}{(I+B)\log(I+B^{-1})}.$$

By easy calculations, we have

$$\frac{I}{(I+A)n\{(I+A^{-1})^{\frac{1}{n}} - I\}} = \frac{I+A^{-1}-I}{(I+A^{-1})n\{(I+A^{-1})^{\frac{1}{n}} - I\}}$$

$$= \frac{\{(I+A^{-1})^{\frac{1}{n}} - I\}\{(I+A^{-1})^{1-\frac{1}{n}} + (I+A^{-1})^{1-\frac{2}{n}} + \ldots + (I+A^{-1})^{\frac{1}{n}} + I\}}{(I+A^{-1})n\{(I+A^{-1})^{\frac{1}{n}} - I\}} \quad \text{by (1.1)}$$

$$= \frac{1}{n}\{(I+A^{-1})^{-\frac{1}{n}} + (I+A^{-1})^{-\frac{2}{n}} + \ldots + (I+A^{-1})^{\frac{1}{n}-1} + (I+A^{-1})^{-1}\}$$

$$\geq \frac{1}{n}\{(I+B^{-1})^{-\frac{1}{n}} + (I+B^{-1})^{-\frac{2}{n}} + \ldots + (I+B^{-1})^{\frac{1}{n}-1} + (I+B^{-1})^{-1}\} \quad \text{by (LH-2)}$$

$$= \frac{I}{(I+B)n\{(I+B^{-1})^{\frac{1}{n}} - I\}}$$

and tending $n \to \infty$, we have (2.1) by (1.2), so the proof of (i) is complete.

(ii). By the same way as (i), we have (ii).

By the same way as the proof of Theorem 2.1, we have the following results and we omit the complete proofs.

Theorem 2.2
(i). $f(t) = \dfrac{t-1-\log t}{\log^2 t}$ *is an operator monotone function.*
(ii). $g(t) = \dfrac{t\log^2 t}{t-1-\log t}$ *is an operator monotone function.*

Remark 2.1. Let $f(t)$ be a continuous function $(0,\infty) \to (0,\infty)$. It is known that $f(t)$ is an operator monotone if and only if $g(t) = \dfrac{t}{f(t)} = f^*(t)$ is also an operator monotone (for example,[5][8][9]), (i) is equivalent to (ii) in Theorem 2.1, here we can give direct and elementary proofs of (i) and (ii) respectively. Although several examples of operator monotone functions are shown in [9], we state an elementary method to construct concrete examples of operator monotone functions by only applying Theorem LH without appealing to Theorem L.

Theorem 2.3. $f(t) = \dfrac{t(t+2)}{(t+1)^2}\log(t+2)$ *is an operator monotone function.*

Theorem 2.4. $f(t) = \dfrac{t(t+1)}{(t+2)\log(t+2)}$ *is an operator monotone function.*

Corollary 2.5.
(i) $f(t) = \dfrac{(t^2-1)\log(1+t)}{t^2}$ *is an operator monotone function.*
(ii) $g(t) = \dfrac{t(t-1)}{(t+1)\log(1+t)}$ *is an operator monotone function.*

§3. Elementary proof of the result that $f_p(t) = \frac{p-1}{p}\left(\frac{t^p-1}{t^{p-1}-1}\right)$ is operator monotone for $-1 \le p \le 2$ by only using Löwner-Heinz inequality

The following Theorem A is shown in [4] by using Bendant-Sharman theorem [1] and also Theorem A is shown in [7] by using Pick functions closely related to Theorem L, and we shall give an elementary proof of Theorem A by only applying Löwner-Heinz inequality without appealing to Theorem L.

Theorem A. $f_p(t) = \dfrac{p-1}{p}\left(\dfrac{t^p-1}{t^{p-1}-1}\right)$ *is an operator monotone function for* $-1 \le p \le 2$.

$f_p(t)$ in Theorem A contains several useful means, for example,

$$f_2(t) = \frac{t+1}{2} \text{ (arithmetic mean)}$$

$$f_1(t) = \frac{t-1}{\log t} \text{ (logarithmic mean)}$$

$$f_{\frac{1}{2}}(t) = \sqrt{t} \text{ (geometric mean)}$$

and

$$f_{-1}(t) = \frac{2}{t^{-1}+1} \text{ (harmonic mean)}$$

At first we state the following fundamental result.

Proposition 3.1. $g_p(t) = \dfrac{t-1}{t^p-1}$ *is an operator monotone function for* $p \in (0,1]$.

Proof. We have only to prove the result for $p = \dfrac{k}{n}$ for natural numbers n and k such that $n \geq k \geq 1$ by continuity of an operator.

$$g_p(t) = \frac{t-1}{t^{\frac{k}{n}}-1} = \frac{(t^{\frac{1}{n}}-1)(t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+\ldots+t^{\frac{k}{n}}+t^{\frac{k-1}{n}}+\ldots t^{\frac{1}{n}}+1)}{(t^{\frac{1}{n}}-1)(t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+\ldots+t^{\frac{1}{n}}+1)}$$

$$(*)\quad = 1 + \frac{t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+\ldots+t^{\frac{k}{n}}}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+\ldots+t^{\frac{1}{n}}+1}$$

$$= 1 + \frac{1}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+\ldots+t^{\frac{1}{n}}+1}\sum_{l=1}^{n-k} t^{\frac{n-l}{n}}$$

$$= 1 + \left(t^{\frac{-(n-k)}{n}}+t^{\frac{-(n-k+1)}{n}}+\ldots+t^{\frac{-(n-1)}{n}}\right)^{-1} + \left(t^{\frac{-(n-k-1)}{n}}+t^{\frac{-(n-k)}{n}}+\ldots+t^{\frac{-(n-2)}{n}}\right)^{-1}$$

$$+\ldots+\left(t^{\frac{-1}{n}}+t^{\frac{-2}{n}}+\ldots+t^{\frac{-k}{n}}\right)^{-1}$$

so that $g_p(t)$ is an operator monotone function by (LH-1). □

For the proof of Theorem A, it suffices to prove the result for all rational numbers $p \in [-1,2]$ by continuity of an operator by using Proposition 3.1. We omit its proof.

We remark that $f_{\frac{1}{2}-d}(t)$ and $f_{\frac{1}{2}+d}(t)$ are both operator monotone for $0 \leq d \leq \frac{3}{2}$ by Theorem A and it is easily verified that $f_{\frac{1}{2}-d}(t) = \dfrac{t}{f_{\frac{1}{2}+d}(t)}$ holds.

The complete version of this paper will appear elsewhere with proofs.

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